Continuous dependence on modelling for temperature dependent bidispersive flow

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Abstract

We consider a model for flow in a porous medium which has a double porosity structure. There is the usual porosity herein called macro porosity, but in addition, we allow for a porosity due to cracks or fissures in the solid skeleton. The cracks give rise to a micro porosity. The model considered also allows for temperature effects with a single temperature T. This paper analyzes three aspects of structural stability. The first establishes continuous dependence of the solution on the interaction coefficient between the velocities associated to the macro and micro porosity. The second analyses continuous dependence on the viscosity coefficients while the third establishes continuous dependence upon the radiation constant when Newton's law of cooling is involved on the boundary.

1 Introduction

A bidispersive (or double porosity) medium is a porous body which has the usual macro pores, but there are also cracks or fissures in the solid skeleton which give rise to micro pores. The macro porosity ϕ is the ratio of the volume of the macro pores to the total volume of the saturated porous material. In addition, the smaller micro pores, see e.g. the picture on page 3069 of Nield and Kuznetsov [1], give rise to a micro porosity ϵ which is the ratio of the volume occupied by the micro pores to the volume of the porous material which remains once the macro pores have been removed. Theories for bidispersive porous materials were introduced in the late 1990s, see e.g. Nield [2], [3], Nield and Bejan [4], Nield and Kuznetsov [1], and see also chapter 13 of Straughan [5].

Inclusion of temperature in bidispersive flows is very important since thermal effects may induce cracking in the solid skeleton which in turn leads to micro pores, see Gelet *et al.* [6], Kim & Hosseini [7]. Applications of bidispersive flow in porous media include oil reservoir recovery, see e.g. Olusola *et al.* [8], and landslides with their catastrophic effect on human life, see e.g. Borja *et al.* [9], Montrasio *et al.* [10], Pooley [11], Scotto di Santolo and Evangelista [12] Several other important applications are described in the books by Straughan [5, 13].

Hirsch and Smale [14], p. 304, pose the problem of what effect does changing the differential equation have upon the solution. They argue that this is the problem of structural stability. They formally define structural stability for a set of differential equations in $\mathbb{R}^n$ on page 312 of Hirsch and Smale [14]. Basically, they argue that a set of equilibrium points or periodic attractors (orbits) should be preserved under small perturbations of the equations themselves, although the actual equilibrium positions or periods could change, and this gives rise to the phenomenon of structural stability. However, for $n = 3$ and for certain parameter values the Lorenz equations possess a chaotic attractor, see Hirsch *et al.* [15], pp. 305–328. Hirsch & Smale [14], pp. 320–321, observe that even for structurally stable systems there may be limit sets which are extremely complicated. Structural stability for the geometric Lorenz attractor is established by Guckenheimer & Williams [16]. Further exotic structures are discussed in detail by Hirsch *et al.* [15] in chapters 15 and 16. Closely connected to structural stability is the phenomenon of continuous dependence on the model itself, although one has to be precise about the type of continuity and the measure in which the analysis is achieved. To describe continuous dependence on the model we suppose we have a set of partial differential equations together with boundary and initial conditions. The equations and boundary conditions contain a term or parameter f and we denote the solution by u. Consider the system with f replaced by f_1 and f_2 and corresponding solutions u_1 and u_2. The solution depends continuously on the model if we may establish a relation of form $m(u_1 - u_2) \le c\rho(f_1 - f_2)$ where c is a constant and m and ρ are suitable positive measures. Thus, roughly, a small change in the equations or boundary conditions, manifest by a change in f, results in a suitably small change in the solution u. Some writers use the expression structural stability synonymously with continuous dependence on modelling but since the present writers have no information on the limit set for our partial differential equations, (2), and our estimates are not valid in the limit t tends to infinity, we shall employ the concept of continuous dependence on the model. One of the reasons to establish continuous dependence on the model itself is that it *may* assist in determining parameter ranges where complicated structures like chaotic attractors arise.

Structural stability and continuous dependence on the model itself, are very important as highlighted in the books of Hirsch and Smale [14], Bellomo and Preziosi [17] and Flavin and Rionero [18]. Within the field of elasticity continuous dependence on modelling has been comprehensively analyzed by Knops and Payne [19, 20].

We draw attention to the fact that structural stability is currently an important topic in the mathematical literature and analyses of this nature in various fields of continuum mechanics may be found in the recent works of Ames and Hughes [21], Castro *et al.* [22], Celebi and Kalantarov [23], Celik and Hoang

[24], Chirita *et al.* [25], Eltayeb [26], Harfash [27], Hill et al. [28], Liu [29], Markowich et al. [30], Otani and Uchida [31], Straughan [13] and Varsakelis and Papalexandris [32].

Our specific goal in this paper is to establish continuous dependence on modelling results for a system of partial differential equations which describe non-isothermal fluid flow in a bidispersive porous material. We firstly demonstrate continuous dependence of the solution upon the interaction parameter for flow between the macro and micro pores. Next, we study continuous dependence on the viscosity coefficients and finally we analyze a thermal boundary condition and establish continuous dependence of the solution upon the coefficient of Newton cooling.

2 Equations of motion

Equations for thermal convection in a bidispersive porous medium, with a single temperature, T, were derived by Falsaperla et al. [33]. These authors deduced the equations by appealing to the general theory of Nield [2, 3] and Nield and Kuznetsov [1]. The equations of Falsaperla *et al.* [33] have form

$$\begin{aligned} &\frac{\mu}{K_f}\,U_i^f + \zeta(U_i^f - U_i^p) = -p^f_{,i} + \rho_F g\alpha T k_i,\\ &\frac{\mu}{K_p}\,U_i^p - \zeta(U_i^f - U_i^p) = -p^p_{,i} + \rho_F g\alpha T k_i,\\ &(\rho c)_m T_{,t} + (\rho c)_f (U_i^f + U_i^p) T_{,i} = \kappa_m \Delta T. \end{aligned} \tag{1}$$

In these equations f and p refer to the macro and micro porous quantities, μ is the dynamic viscosity of the saturating fluid, K_f and K_p are permeabilities, U_i^f and U_i^p are fluid velocities, p^f and p^p are pressures, ζ is an interaction coefficient, g is gravity, α is the coefficient of thermal expansion of the fluid, ρ_F is a reference temperature, T is the temperature of the fluid, $(\rho c)_m$ and $(\rho c)_f$ are the products of the density and specific heat at constant pressure with the m indicating an averaged value over the bidispersive porous medium while the f indicates the fluid itself, these quantities being defined precisely in Falsaperla *et al.* [33]. Furthermore, κ_m is the averaged thermal conductivity of the bidispersive porous medium defined in Falsaperla *et al.* [33] and $\mathbf{k} = (0, 0, 1)$. Throughout, standard indicial notation will be employed together with the Einstein summation convention, and Δ is the Laplace operator. Equations (1) represent momentum balance in the macro and micro pores where a Bousinesq approximation has been employed to yield the buoyancy terms linear in temperature. The final equation in (1) is the balance of energy as derived in Falsaperla *et al.* [33]. The Boussinesq approximation is derived under the assumptions that $|\alpha(T - T_R)| << 1$ where T_R is a reference temperature, and the velocity gradients are suitably small so the viscous dissipation may be neglected in equation $(1)_3$, cf. Straughan [5], pp. 16–21, or Roberts [34], pp. 196–197. This approximation has been the subject of many recent articles and further details may be found in Barletta [35], Feireisl & Novotny [36], Gouin & Ruggeri [37], Gouin *et al.* [38], Nield & Barletta [39], Rajagopal *et al.* [40], Rajagopal *et al.* [41, 42]. The restriction on the temperature and on the velocity gradients limits the physical domain of validity of equations (1) and subsequently equations (2).

Without loss of generality for the issue of continuous dependence upon modelling under investigation in this work we simplify equations (1) as follows. We replace U_i^f and U_i^p by u_i and v_i, we substitute μ/K_f and μ/K_p by μ and γ, and we denote $\rho_F g\alpha k_i$ by g_i. We further divide $(1)_3$ by κ_m and put $\alpha = (\rho c)_f/\kappa_m$. Next, rescale time so that the coefficient of $T_{,t}$ is replaced by the value one. In this manner the equations of flow in a bidispersive porous medium may be taken to be

$$\begin{aligned}
&\mu u_i + \zeta(u_i - v_i) = -p_{,i} + g_i T\,, \quad u_{i,i} = 0\,,\\
&\gamma v_i - \zeta(u_i - v_i) = -q_{,i} + g_i T\,, \quad v_{i,i} = 0\,,\\
&T_{,t} + \alpha(u_i + v_i)T_{,i} = \Delta T\,.
\end{aligned} \tag{2}$$

We suppose equations (2) are defined on a bounded domain $\Omega \subset \mathbb{R}^3$, with boundary Γ which is sufficiently smooth to allow application of the divergence theorem. The conditions on the velocities on the boundary are

$$u_i n_i = 0\,, \quad v_i n_i = 0\,, \qquad \mathbf{x} \in \Gamma\,,\ t > 0, \tag{3}$$

where n_i represents the unit outward normal to Γ. The initial condition is

$$T = T_0(\mathbf{x})\,, \quad \mathbf{x} \in \Omega. \tag{4}$$

Without loss of generality for the class of problems under consideration we suppose the gravity vector is bounded as

$$|\mathbf{g}| \le 1\,.$$

(We could replace the value of 1 by a general bound $M < \infty$, or divide equations $(2)_{1,2}$ by $\rho_F g\alpha$ and redefine the remaining coefficients.) We let $\|\cdot\|$ and $(\cdot,\cdot)$ be the norm and the inner product on $L^2(\Omega)$.

While one may examine many aspects of continuous dependence on modelling for equations (2) we believe this is the first such study and so we restrict attention to analysing continuous dependence upon the interaction coefficient ζ and on the coefficients μ and γ. Since the latter coefficients may in general depend on temperature a continuous dependence study should prove beneficial. However, continuous dependence on modelling encompasses all aspects of the model and not just the coefficients of the equations. For example, it is important to know information on continuous dependence on the boundary conditions, continuous dependence on the geometry of the domain, and continuous dependence upon the initial-time geometry to assess whether one can be sure one has measured initial data at the same time or whether one should allow variation of specification of initial data over a continuous series of times known as the initial-time geometry, see e.g. Payne & Straughan [43, 44]. To incorporate continuous dependence on an aspect of modelling rather than a coefficient in the equations we here include an anlysis of continuous dependence on modelling the boundary conditions. We specifically analyse continuous dependence on the Newton cooling coefficient when a combination of temperature and heat flux is given on the boundary.

3 Continuous dependence on the interaction coefficient

We now investigate continuous dependence on the interaction coefficient ζ. In addition to (3) we suppose T is known on the boundary, i.e.

$$T = h(\mathbf{x}, t)\,, \qquad \text{on } \Gamma \times \{t > 0\}\,, \tag{5}$$

where h is a prescribed function. We point out that existence of a strong solution to the analogous problem to (2) - (4) and (5) for the single porosity case has been established by Castro *et al.* [45]. We believe their methods may be extended to the problem in hand to establish existence of a strong solution here. It is worth pointing out that Castro *et al.* [45] establish existence for a class of fractional diffusions for the temperature field but their work does include the Laplacian of T as in $(2)_3$.

To investigate continuous dependence on ζ we let $(u_{1\,i}, v_{1\,i}, p_1, q_1, T_1)$ and $(u_{2\,i}, v_{2\,i}, p_2, q_2, T_2)$ be solutions to (2)-(4) for the same coefficients and boundary data, excepting $(u_{1\,i}, v_{1\,i}, p_1, q_1, T_1)$ is the solution when ζ has the value ζ_1 whereas $(u_{2\,i}, v_{2\,i}, p_2, q_2, T_2)$ is the solution for ζ having value ζ_2.

Define the variables $(w_i, r_i, \pi, \xi, \theta)$ and β by

$$\begin{aligned} w_i = u_{1\,i} - u_{2\,i}\,, \quad r_i = v_{1\,i} - v_{2\,i}\,, \quad \pi = p_1 - p_2\,, \\ \xi = q_1 - q_2\,, \quad \theta = T_1 - T_2\,, \quad \beta = \zeta_1 - \zeta_2\,. \end{aligned} \tag{6}$$

By subtraction we derive the difference equations from (2) as

$$\begin{aligned} &\mu w_i + \zeta_1(w_i - r_i) + \beta(u_{2\,i} - v_{2\,i}) = -\pi_{,i} + g_i\theta\,, \quad u_{i,i} = 0\,, \\ &\gamma r_i - \zeta_1(w_i - r_i) - \beta(u_{2\,i} - v_{2\,i}) = -\xi_{,i} + g_i\theta\,, \quad v_{i,i} = 0\,, \\ &\theta_{,t} + \alpha(w_i + r_i)T_{1,\,i} + \alpha(u_{2\,i} + v_{2\,i})\theta_{,\,i} = \Delta\theta\,, \end{aligned} \tag{7}$$

in $\Omega \times (0, \mathcal{T})$, some $\mathcal{T} < \infty$.

The relevant difference boundary conditions are

$$w_i n_i = 0\,, \quad r_i n_i = 0\,, \quad \theta = 0\,, \qquad \mathbf{x} \in \Gamma\,,\, t \in (0, \mathcal{T}]. \tag{8}$$

To proceed, we multiply $(7)_1$ by w_i and integrate over Ω and then multiply $(7)_2$ by r_i before integrating over Ω. After addition of the results we find

$$\mu\|\mathbf{w}\|^2 + \gamma\|\mathbf{r}\|^2 + \zeta_1\|\mathbf{w} - \mathbf{r}\|^2 = -\beta(u_{2\,i} - v_{2\,i}, w_i - r_i) + (g_i\theta, w_i + r_i)\,. \tag{9}$$

Let $k = \mu^{-1} + \gamma^{-1}$ and then upon employing the arithmetic-geometric mean inequality on the terms on the right of (9) we may obtain

$$\begin{aligned} -\beta(u_{2i} - v_{2i}, w_i - r_i) + (g_i\theta, w_i + r_i) \leq & \frac{1}{4\zeta_1}\beta^2\|\mathbf{u}_2 - \mathbf{v}_2\|^2 + \zeta_1\|\mathbf{w} - \mathbf{r}\|^2 \\ & + \frac{k}{2}\|\theta\|^2 + \frac{\mu}{2}\|\mathbf{w}\|^2 + \frac{\gamma}{2}\|\mathbf{r}\|^2, \end{aligned} \tag{10}$$

and use of this in (9) leads to

$$\mu\|\mathbf{w}\|^2 + \gamma\|\mathbf{r}\|^2 \leq k\|\theta\|^2 + \frac{\|\mathbf{u}_2 - \mathbf{v}_2\|^2}{2\zeta_1}\beta^2\,. \tag{11}$$

Next, multiply $(7)_3$ by θ and integrate over Ω. After some integration by parts one may see that

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = \alpha(T_1 w_i, \theta,_i) + \alpha(T_1 r_i, \theta,_i) - \|\nabla\theta\|^2\,. \tag{12}$$

We require an a priori estimate for T_1. To this end define

$$T_m = max\{\|T_0\|_\infty, \sup_{[0,\mathcal{T}]}|h|\}\,.$$

Then one may employ the function

$$\psi = [T_1 - T_m]^+ = sup(T_1 - T_m, 0)$$

and generalize the proof in Payne et al. [46], pp. 432-433, to show that

$$\sup_{\Omega\times[0,\mathcal{T}]} T_1(\mathbf{x}, t) \leq T_m\,. \tag{13}$$

This allows us to deduce from (12)

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \leq \alpha T_m\|\mathbf{w}\|\|\nabla\theta\| + \alpha T_m\|\mathbf{r}\|\|\nabla\theta\| - \|\nabla\theta\|^2\,. \tag{14}$$

Now, employ the arithmetic-geometric mean inequality on the terms involving $\|\mathbf{w}\|$ and $\|\mathbf{r}\|$. In this way we find

$$\frac{d}{dt}\|\theta\|^2 \leq k_1(\|\mathbf{w}\|^2 + \|\mathbf{r}\|^2)\,, \tag{15}$$

where $k_1 = (\alpha T_m)^2$.

To use inequality (11) we now need to bound $\|\mathbf{u}_2 - \mathbf{v}_2\|$. Multiply equations $(2)_{1,2}$ evaluated for $u_{2\,i}$ and $v_{2\,i}$ by $u_{2\,i}$ and $v_{2\,i}$, respectively. After integration by parts this leads to

$$\mu\|\mathbf{u}_2\|^2 + \gamma\|\mathbf{v}_2\|^2 + \zeta_2\|\mathbf{u}_2 - \mathbf{v}_2\|^2 = (g_i T_2, u_{2\,i}) + (g_i T_2, v_{2\,i})\,.$$

Now use the arithmetic-geometric mean inequality to deduce

$$\zeta_2\|\mathbf{u}_2-\mathbf{v}_2\|^2\le\frac{k}{4}\|T_2\|^2\,. \tag{16}$$

We may apply the same argument to that above to deduce $T_2\le T_m$ and then (16) allows us to see that

$$\|\mathbf{u}_2-\mathbf{v}_2\|^2\le\frac{k}{4\zeta_2}T_m^2|\Omega|\,, \tag{17}$$

where $|\Omega|$ is the measure of Ω. Hence, from (11) we obtain

$$\mu\|\mathbf{w}\|^2+\gamma\|\mathbf{r}\|^2\le k\|\theta\|^2+\frac{k}{8\zeta_1\zeta_2}T_m^2|\Omega|\beta^2\,. \tag{18}$$

By employing (18) in inequality (15) we now obtain

$$\frac{d}{dt}\|\theta\|^2\le k_2\|\theta\|^2+k_3\beta^2\,, \tag{19}$$

where

$$k_2=k_1k^2,\qquad k_3=\frac{k^2T_m^2|\Omega|}{8\zeta_1\zeta_2}\,.$$

Inequality (19) integrates to yield

$$\|\theta(t)\|^2\le B\beta^2\,,\qquad \forall t\in[0,\mathcal{T}]\,, \tag{20}$$

where $B=k_3[\exp(k_2t)-1]/k_2$. Thus, (20) demonstrates continuous dependence of θ upon the interaction coefficient β. We then obtain continuous dependence upon β for w_i and r_i by using (20) in (18). Note the fact that B is a function of t is typical in continuous dependence analysis, see e.g. Hirsch *et al.* [15], pp. 394–397.

4 Continuous dependence on the viscosity coefficients

In this section we establish a continuous dependence estimate for the solution to (2) upon the viscosity coefficients μ and γ. Thus, we let $\mathbf{U}_1=(u_{1i},v_{1i},p_1,q_1,T_1)$ and $\mathbf{U}_2=(u_{2i},v_{2i},p_2,q_2,T_2)$ be solutions to (2) - (4) and (5) for the same functions T_0 and h but $\mathbf{U}_1$ and $\mathbf{U}_2$ satisfy (2) for different coefficients μ_1,γ_1 and μ_2,γ_2.

Define the difference variables (w_i,r_i,π,ξ,θ) as in (6) but now define δ and λ by

$$\delta=\mu_1-\mu_2,\qquad \lambda=\gamma_1-\gamma_2. \tag{21}$$

By subtraction we now find that the difference variables satisfy the equations

$$\begin{aligned}&\mu_1w_i+\delta u_{2i}+\zeta(w_i-r_i)=-\pi_{,i}+g_i\theta,\\&\gamma_1r_i+\lambda v_{2i}-\zeta(w_i-r_i)=-\omega_{,i}+g_i\theta,\\&\theta_{,t}+\alpha(w_i+r_i)T_{1,i}+\alpha(u_{2i}+v_{2i})\theta_{,i}=\Delta\theta.\end{aligned} \tag{22}$$

The boundary conditions are

$$w_i n_i = 0, \quad r_i n_i = 0, \quad \theta = 0, \qquad \mathbf{x} \in \Gamma,\ t \in (0, \mathcal{T}], \tag{23}$$

while the initial condition is

$$\theta(\mathbf{x}, 0) = 0, \qquad \mathbf{x} \in \Omega. \tag{24}$$

Multiply $(22)_1$ by w_i and integrate over Ω, likewise multiply $(22)_2$ by r_i and integrate over Ω, and employ the boundary conditions to obtain

$$\mu_1 \|\mathbf{w}\|^2 + \gamma_1 \|\mathbf{r}\|^2 + \zeta \|\mathbf{w} - \mathbf{r}\|^2 = (g_i \theta, w_i + r_i) - \delta(u_{2i}, w_i) - \lambda(v_{2i}, r_i). \tag{25}$$

Multiply $(22)_3$ by θ and integrate over Ω and after use of the boundary conditions we may obtain

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\theta\|^2 + \|\nabla\theta\|^2 =& \alpha(w_i + r_i, T_1 \theta_{,i}) \\ \leq& \alpha T_m (\|\mathbf{w}\| \, \|\nabla\theta\| + \|\mathbf{r}\| \, \|\nabla\theta\|), \end{aligned} \tag{26}$$

where in the last step we have employed (13). We now employ the arithmetic-geometric mean inequality on the right hand side of (25) and discard the ζ term to derive

$$\frac{\mu_1}{2} \|\mathbf{w}\|^2 + \frac{\gamma_1}{2} \|\mathbf{r}\|^2 \leq c \|\theta\|^2 + \frac{1}{\mu_1} \delta^2 \|\mathbf{u}_2\|^2 + \frac{1}{\gamma_1} \lambda^2 \|\mathbf{v}_2\|^2, \tag{27}$$

where $c = 1/\mu_1 + 1/\gamma_1$. Next, use the arithmetic-geometric mean inequality on the right of (26) and balance the resulting $\|\nabla\theta\|^2$ terms by those on the left to see that

$$\frac{d}{dt} \|\theta\|^2 \leq (\alpha T_m)^2 (\|\mathbf{w}\|^2 + \|\mathbf{r}\|^2). \tag{28}$$

Put now

$$\tilde{\mu} = \min \left\{ \frac{\mu_1}{2}, \frac{\gamma_1}{2} \right\}$$

and then from (27) and (28) we may show

$$\frac{d}{dt} \|\theta\|^2 \leq c_1 c \|\theta\|^2 + \frac{c_1}{\mu_1} \delta^2 \|\mathbf{u}_2\|^2 + \frac{c_1}{\gamma_1} \lambda^2 \|\mathbf{v}_2\|^2, \tag{29}$$

where $c_1 = (\alpha T_m)^2 / \tilde{\mu}$. We must now derive *a priori* bounds for $\|\mathbf{u}_2\|$ and $\|\mathbf{v}_2\|$.

Multiply equations $(2)_1$ and $(2)_2$ defined for the two variables u_{2i} and v_{2i}, respectively, and integrate over Ω to deduce

$$\mu_2 \|\mathbf{u}_2\|^2 + \gamma_2 \|\mathbf{v}_2\|^2 + \zeta \|\mathbf{u}_2 - \mathbf{v}_2\|^2 = g_i(T_2, u_{2i} + v_{2i}). \tag{30}$$

Next, employ the arithmetic-geometric mean inequality on the right of (30) and discard the ζ term to find

$$\mu_2 \|\mathbf{u}_2\|^2 + \gamma_2 \|\mathbf{v}_2\|^2 \leq c_2 \|T_2\|^2, \tag{31}$$

where $c_2 = 1/\mu_2 + 1/\gamma_2$. The argument leading to (13) applies also to T_2 and so from (31) we may obtain

$$\mu_2 \|\mathbf{u}_2\|^2 + \gamma_2 \|\mathbf{v}_2\|^2 \leq c_2 |\Omega| T_m^2 \,. \tag{32}$$

Put $\hat{\mu} = \min\{\mu_2, \gamma_2\}$ and return to inequality (29) and employ (32) to see that

$$\frac{d}{dt} \|\theta\|^2 \leq \ell_1 \|\theta\|^2 + \ell_2 \delta^2 + \ell_3 \lambda^2, \tag{33}$$

where

$$\ell_1 = c_1 c, \qquad \ell_2 = \frac{c_1 c_2}{\mu_1 \hat{\mu}} \, T_m^2 |\Omega|, \qquad \ell_3 = \frac{c_1 c_2}{\lambda_1 \hat{\mu}} \, T_m^2 |\Omega|.$$

This inequality is integrated to obtain

$$\|\theta(t)\|^2 \leq C(\ell_2 \delta^2 + \ell_3 \lambda^2), \qquad \forall t \in [0, \mathcal{T}], \tag{34}$$

where $C = [\exp(\ell_1 t) - 1]/\ell_1$.

Inequality (34) represents a continuous dependence on modelling estimate for $\theta(\mathbf{x}, t)$ on the viscosity coefficients. To obtain an analogous estimate for w_i and r_i we use inequality (27) together with (32) to derive

$$\|\mathbf{w}\|^2 + \|\mathbf{r}\|^2 \leq \frac{2c}{\tilde{\mu}} \, \|\theta\|^2 + c_3 \delta^2 + c_4 \lambda^2, \tag{35}$$

where

$$c_3 = \frac{2c_2}{\mu_1 \tilde{\mu} \hat{\mu}} \, |\Omega| T_m^2, \qquad c_4 = \frac{2c_2}{\gamma_1 \tilde{\mu} \hat{\mu}} \, |\Omega| T_m^2 \,.$$

By combining (34) and (35) one shows

$$\|\mathbf{w}(t)\|^2 + \|\mathbf{r}(t)\|^2 \leq \ell_4 \delta^2 + \ell_5 \lambda^2, \tag{36}$$

where

$$\ell_4 = \frac{2cC\ell_2}{\tilde{\mu}} + c_3, \qquad \ell_5 = \frac{2cC\ell_3}{\tilde{\mu}} + c_4 \,.$$

Thus, continuous dependence on modelling for the viscosity coefficients is proved in the sense of (34) and (36).

5 Continuous dependence on the Newton cooling coefficient

Equation (5) is a Dirichlet condition on the temperature field on the boundary Γ. We now wish to consider an aspect of continuous dependence upon modelling involving the boundary conditions. In general, rather than prescribing the temperature on the boundary one might give a combination of the temperature and heat flux. This yields a general boundary condition of form

$$\frac{\partial T}{\partial n} + \kappa T = F(\mathbf{x}, t), \qquad \text{on } \Gamma \times \{t > 0\}, \tag{37}$$

for F a given function and for $\kappa > 0$ a constant. We consider continuous dependence of the solution upon the parameter κ in the case where $\kappa T_a = F$, with T_a being physically the ambient temperature outside of the porous body. Equation (37) gives rise to a condition of Newton cooling. Thus, instead of the boundary conditions (3) and (5) we consider in this section boundary conditions of type

$$u_i n_i = 0\,, \quad v_i n_i = 0 \qquad \text{on } \Gamma \times \{t > 0\}\,, \tag{38}$$

and

$$\frac{\partial T}{\partial n} = -\kappa[T - T_a(\mathbf{x}, t)]\,, \quad \text{on } \Gamma \times \{t > 0\}\,, \tag{39}$$

where $\kappa\,(\geq 0)$ is the coefficient of Newton cooling. In (38) n_i is the unit outward normal to Γ, and in equation (39) $\partial/\partial n$ is the unit outward normal derivative. The initial condition is again (4).

To study continuous dependence on κ we let $(u_{1_i}, v_{1_i}, p_1, q_1, T_1)$ and $(u_{2_i}, v_{2_i}, p_2, q_2, T_2)$ be solutions to (2), (4), (38), (39) for the same coefficients $\mu, \gamma, \zeta, \alpha$ and T_a, but for different cooling parameters κ_1 and κ_2, respectively.

Define the difference variables w_i, r_i, π, ξ and θ as in (6) and define the variable ν by

$$\nu = \kappa_1 - \kappa_2\,. \tag{40}$$

One in this case determines the difference equations as

$$\begin{aligned} &\mu w_i + \zeta(w_i - r_i) = -\pi,_i + g_i\theta\,,\\ &\gamma r_i - \zeta(w_i - r_i) = -\xi,_i + g_i\theta\,,\\ &\theta,_t + \alpha(u_{1\,i} + v_{1\,i})\theta,_i + \alpha(w_i + r_i)T_{2,\,i} = \Delta\theta \end{aligned} \tag{41}$$

in $\Omega \times (0, \mathcal{T})$, some $\mathcal{T} < \infty$.

The boundary and initial conditions are

$$\begin{aligned} w_i n_i = 0\,, \quad r_i n_i = 0\,,\\ \frac{\partial\theta}{\partial n} = -\kappa_1\theta - \nu(T_2 - T_a)\,, \end{aligned} \tag{42}$$

on $\Gamma \times (0, \mathcal{T}]$, and

$$\theta(\mathbf{x}, 0) = 0\,, \qquad \mathbf{x} \in \Omega\,. \tag{43}$$

The proof of continuous dependence begins by multiplying $(41)_1$ by w_i, $(41)_2$ by r_i, integrating over Ω and adding to find

$$\mu\|\mathbf{w}\|^2 + \gamma\|\mathbf{r}\|^2 + \zeta\|\mathbf{w} - \mathbf{r}\|^2 = (g_i\theta, w_i + r_i)\,. \tag{44}$$

We employ the Cauchy-Schwarz and arithmetic-geometric mean inequalities on the right hand side of (44) to obtain

$$\mu\|\mathbf{w}\|^2 + \gamma\|\mathbf{r}\|^2 + 2\zeta\|\mathbf{w} - \mathbf{r}\|^2 \leq k\|\theta\|^2\,. \tag{45}$$

Next, multiply $(41)_3$ by θ and integrate over Ω. After integration by parts and use of the boundary conditions we find

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \alpha(T_2 w_i, \theta,_i) + \alpha(T_2 r_i, \theta,_i) - \|\nabla\theta\|^2 - \kappa_1 \oint_\Gamma \theta^2 dA - \nu \oint_\Gamma \theta(T_2 - T_a)\, dA\,. \tag{46}$$

Upon using the arithmetic-geometric mean inequality on the last term in (46) one obtains

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \leq \alpha(T_2 w_i, \theta,_i) + \alpha(T_2 r_i, \theta,_i) - \|\nabla\theta\|^2 + \frac{\nu^2}{4\kappa_1}\oint_\Gamma (T_2 - T_a)^2 dA\,. \tag{47}$$

The proof of the bound for T_2 in section 3 does not carry over here due to the boundary conditions. Hence, we follow an argument in Payne and Straughan [47].

For $p > 1$ (with $T > 0$, otherwise take p to be an even integer) we see that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega T_2^p\, dx = p\int_\Omega T_2^{p-1} T_{2,\,t}\, dx \\ =&p\int_\Omega T_2^{p-1}\left[\Delta T_2 - \alpha(u_{2\,i} + v_{2\,i})T_{2,i}\right]\, dx \\ =&-p(p-1)\int_\Omega T_2^{p-2}|\nabla T_2|^2\, dx - \kappa_2 p\oint_\Gamma T_2^{p-1}(T_2 - T_a)\, dA\,. \end{aligned} \tag{48}$$

Young's inequality is used on the last term on the right of (48) to find

$$\frac{d}{dt}\int_\Omega T_2^p\, dx \leq -p(p-1)\int_\Omega T_2^{p-2}|\nabla T_2|^2\, dx + \kappa_2\left(\frac{p-1}{p}\right)^{p-1}\oint_\Gamma T_a^p\, dA\,. \tag{49}$$

Upon integration one may deduce from (49)

$$\left[\int_\Omega T_2^p dx\right]^{\frac{1}{p}} \leq \left[\int_\Omega T_0^p\, dx + \kappa_2\left(\frac{p-1}{p}\right)^{p-1}\int_0^t\oint_\Gamma T_a^p\, dA\, ds\right]^{\frac{1}{p}}\,.$$

Allow $p \to \infty$ to see that

$$\sup_\Omega |T_2| \leq T_m \tag{50}$$

where now

$$T_m = max\{\sup_\Omega |T_0|, \sup_{\Gamma\times[0.\mathcal{T}]} |T_a|\}\,.$$

Next, from (45)

$$\|\mathbf{w}\| \leq a_1\|\theta\|\,, \qquad \|\mathbf{r}\| \leq a_2\|\theta\|\,, \tag{51}$$

where

$$a_1 = \sqrt{\frac{k}{\mu}}\,, \qquad a_2 = \sqrt{\frac{k}{\gamma}}\,.$$

Thus, employing (50) and (51) we find

$$\alpha(T_2 w_i, \theta_{,i}) + \alpha(T_2 r_i, \theta_{,i}) \leq a_3 \|\theta\| \|\nabla\theta\| \,, \tag{52}$$

where

$$a_3 = \alpha T_m a_1 + \alpha T_m a_2 \,.$$

Use (52) in (47) together with the arithmetic-geometric mean inequality to find

$$\frac{d}{dt}\|\theta\|^2 \leq \frac{a_3^2}{2}\|\theta\|^2 + A\nu^2 \,, \tag{53}$$

where

$$A(t) = \frac{1}{2\kappa_1}\oint_\Gamma (T_m - T_a)^2 dA \,,$$

in which we have extended estimate (50) to the boundary Γ by continuity.

Inequality (53) may now be integrated to yield the following continuous dependence on κ inequality

$$\|\theta(t)\|^2 \leq R(t)\nu^2 \,, \tag{54}$$

where R is given by

$$R(t) = \int_0^t A(s) \exp\left[\frac{1}{2}a_3^2(t-s)\right] ds \,.$$

Estimate (45) then allows one to deduce

$$\|\mathbf{w}\|^2 \leq \frac{k}{\mu}R(t)\nu^2 \,, \qquad \|\mathbf{r}\|^2 \leq \frac{k}{\gamma}R(t)\nu^2 \,. \tag{55}$$

To sum up, inequalities (54) and (55) furnish continuous dependence on κ estimates, as required.

6 Conclusions

In this paper we have addressed a model for bidispersive flow in porous media. We have demonstrated continuous dependence on the model by establishing truly a priori estimates for continuous dependence on the interaction coefficient, on the viscosity coefficients, and also on the Newton cooling coefficient. We believe this represents the first mathematical analysis of continuous dependence on modelling in this important field of flow in multi-porosity media, an area which has a multitude of applications in real life, see Straughan [5, 13].

There will be many other aspects to study of continuous dependence on modelling with various models of bidispersive or tridispersive flow, including also different temperatures in the macro and micro pores, as modelled by Nield and Kuznetsov [1] and Kuznetsov and Nield [48], respectively.

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